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On the hydrogen atom via the Wigner–Heisenberg algebra

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Abstract
We extend the usual Kustaanheimo–Stiefel 4D $\rightarrow$ 3D mapping to study and discuss a constrained super-Wigner oscillator in four dimensions. We show that the physical hydrogen atom is the system that emerges in the bosonic sector of the mapped super 3D system.

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1. Introduction

The R-deformed Heisenberg or Wigner–Heisenberg (WH) algebraic technique [1] which was super-realized for quantum oscillators [2–4] is related to the paraboson relations introduced by Green [5].

Let us now point out that the WH algebra is given by the following (anti-)commutation relations ($[A, B]_+ \equiv AB + BA$ and $[A, B]_- \equiv AB - BA$):

$$H = \frac{1}{2}[a^-, a^+]_+, \qquad [H, a^\pm]_- = \pm a^\pm, \qquad [a^-, a^+]_- = 1 + cR, \tag{1}$$

where c is a real constant associated with the Wigner parameter [2] and the R operator satisfies

$$[R, a^\pm]_+ = 0, \qquad R^2 = 1. \tag{2}$$

Note that when $c = 0$, we have the standard Heisenberg algebra.

The generalized quantum condition given in equation (1) has been found to be relevant in the context of integrable models [6]. Furthermore, this algebra was also used to solve the energy eigenvalue and eigenfunctions of the Calogero interaction, in the context of one-dimensional many-body integrable systems, in terms of a new set of phase space variables involving exchanged operators [7, 8]. From this WH algebra, a new kind of deformed calculus has been developed [9–11].

The WH algebra has been considered for the three-dimensional non-canonical oscillator to generate a representation of the orthosympletic Lie superalgebra osp(3/2) [12], and recently Palev *et al* have investigated the 3D Wigner oscillator under a discrete non-commutative context [13, 14]. In addition, the connection of the WH algebra with the Lie superalgebra $s\ell(1|n)$ has been studied in a detailed manner [15].

Recently, the relevance of relations (1) to quantization in fractional dimension has also been discussed [16, 17] and the properties of Weyl-ordered polynomials in operators P and Q, in fractional-dimensional quantum mechanics, have been developed [18].

The Kustaanheimo–Stiefel mapping [19] has been exactly solved and well-studied in the literature. (See for example, Chen [20], Cornish [21], Chen and Kibler [22], D'Hoker and Vinet [23].) Kostelecky, Nieto and Truax [24] have studied in a detailed manner the relation of the supersymmetric (SUSY) Coulombian problem [25–29] in D-dimensions with that of SUSY isotropic oscillators in D-dimensions in the radial version (see also Lahiri *et al* [30]). For the mapping with 3D radial oscillators, see also Bergmann and Frishman [31], Cahill [32] and Chen *et al* [33]. The connection of the D-dimensional hydrogen atom with the D-dimensional harmonic oscillator in terms of the su(1, 1) algebra has been investigated by Zeng *et al* [34]. However, the correspondence mapping of a 4D isotropic constrained super-Wigner oscillator (for super-Wigner oscillators see our previews work [2, 3]) with the corresponding super-system in 3D, such that the usual 3D hydrogen atom emerges in the 4D → 3D mapping in the bosonic sector, has not been studied in the literature; the objectives of the present work are to do such a mapping and to analyze in detail the consequences. In this work, the stationary states of the hydrogen atom are mapped onto the super-Wigner oscillator using the Kustaanheimo–Stiefel transformation.

This work is organized as follows. In section 2, we start by summarizing the R-deformed Heisenberg algebra or the Wigner–Heisenberg algebraic technique for the Wigner oscillator, based on the super-realization of the WH algebra for simpler effective spectral resolutions of general oscillator-related potentials, applied by Jayaraman and Rodrigues, in [2]. In section 3, we illustrate how to construct the 4D → 3D mapping in the bosonic sector which offers a simple resolution of the hydrogen energy spectra and eigenfunctions. The conclusion is given in section 4.

2. The super-Wigner oscillator in 1D

The Wigner oscillator ladder operators

$$a^{\pm} = \frac{1}{\sqrt{2}}(\pm \mathrm{i}\hat{p}_x - \hat{x}) \tag{3}$$

of the WH algebra may be written in terms of the super-realization of the position and momentum operators, namely $\hat{x} = x\Sigma_1$ and $\hat{p}_x = -\mathrm{i}\Sigma_1\frac{\mathrm{d}}{\mathrm{d}x} + \frac{c}{2x}\Sigma_2$, satisfy the general quantum rule $[\hat{x}, \hat{p}_x]_- = \mathrm{i}(1 + cR)$, where $c = 2(\ell + 1)$. Thus, in this representation the reflection operator becomes $R = \Sigma_3$, where Σ_3 is the diagonal Pauli matrix.

Thus, from the super-realized first-order ladder operators given by

$$a^{\pm}(\ell+1) = \frac{1}{\sqrt{2}}\left\{\pm\frac{\mathrm{d}}{\mathrm{d}x} \pm \frac{(\ell+1)}{x}\Sigma_3 - x\right\}\Sigma_1, \qquad \ell > 0, \tag{4}$$

the Wigner Hamiltonian becomes

$$H(\ell+1) = \frac{1}{2}[a^+(\ell+1), a^-(\ell+1)]_+ \tag{5}$$

and the WH algebra ladder relations are readily obtained as

$$[H(\ell+1), a^{\pm}(\ell+1)]_- = \pm a^{\pm}(\ell+1). \tag{6}$$

Equations (5) and (6) together with the commutation relation

$$[a^-(\ell+1), a^+(\ell+1)]_- = 1 + 2(\ell+1)\Sigma_3 \tag{7}$$

constitute the super-WH algebra.

Thus, the super-Wigner oscillator Hamiltonian in terms of Pauli's matrices ($\Sigma_i, i = 1, 2, 3$) is given by

$$\begin{aligned} H(\ell+1) &= \frac{1}{2}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{1}{x^2}(\ell+1)[(\ell+1)\Sigma_3 - 1]\Sigma_3\right\} \\ &= \begin{pmatrix} H_-(\ell) & 0 \\ 0 & H_+(\ell) = H_-(\ell+1) \end{pmatrix}, \end{aligned} \tag{8}$$

where the bosonic and fermionic sector Hamiltonians are respectively given by

$$H_-(\ell) = \frac{1}{2}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{1}{x^2}\ell(\ell+1)\right\} \tag{9}$$

and

$$H_+(\ell) = \frac{1}{2}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{1}{x^2}(\ell+1)(\ell+2)\right\} = H_-(\ell+1). \tag{10}$$

Note that the bosonic sector is the Hamiltonian of the oscillator with a barrier.

The super-Wigner oscillator eigenfunctions that generate the eigenspace associated with even(odd) Σ_3-parity for bosonic(fermionic) quanta $n = 2m (n = 2m + 1)$ are given by

$$\Psi_{n=2m}(x;\ell+1) = \begin{pmatrix} \psi_-^{(m)}(x;\ell) \\ 0 \end{pmatrix}, \qquad \Psi_{n=2m+1}(x;\ell+1) = \begin{pmatrix} 0 \\ \psi_+^{(m)}(x;\ell) \end{pmatrix} \tag{11}$$

and satisfy the following eigenvalue equation:

$$\begin{aligned} &H(\ell+1)\Psi_n(x;\ell+1) = E_n\Psi_n(x;\ell+1) \\ &\Sigma_3\Psi_{n=2m} = \Psi_{n=2m} \\ &\Sigma_3\Psi_{n=2m+1} = -\Psi_{n=2m+1}, \end{aligned} \tag{12}$$

where the non-degenerate energy eigenvalues are obtained by the repeated application of the raising operator on the ground eigenstate

$$\Psi_n(x;\ell+1) \propto (a^+(\ell+1))^n\Psi_0(x;\ell+1) \tag{13}$$

and are given by

$$E_n = \ell + \tfrac{3}{2} + n, \qquad n = 0, 1, 2, \ldots. \tag{14}$$

The ground-state energy eigenfunction satisfies the following annihilation condition:

$$a^-(\ell+1)\Psi_{(0)}(x;\ell+1) = 0, \tag{15}$$

which using equation (4) results in

$$\psi_-^{(0)}(x;\ell) = N_1 x^{(\ell+1)}\,\mathrm{e}^{-\frac{x^2}{2}}, \qquad \psi_+^{(0)}(x;\ell) = N_2 x^{-(\ell+1)}\,\mathrm{e}^{-\frac{x^2}{2}}.$$

If we assume $\ell + 1 > 0$, only $\psi_-^{(0)}(x;\ell)$ meets the physical requirement of vanishing at the origin and $\psi_+^{(0)}(x;\ell)$, which does not stand this test, is discarded by setting $N_2 = 0$ in (15). In this case, the normalizable ground-state eigenfunction is given, up to a normalization constant, by

$$\Psi_0(x;\ell+1) \propto \begin{pmatrix} x^{(\ell+1)}\,\mathrm{e}^{-\frac{1}{2}x^2} \\ 0 \end{pmatrix}, \tag{16}$$

which has even Σ_3-parity, i.e. $\Sigma_3\Psi_0(x;\ell+1) = \Psi_0(x;\ell+1)$.

For the bosonic and fermionic sector Hamiltonians, the energy eigenvectors satisfy the following equation:

$$H_\pm(\ell)\psi_\pm^{(m)}(x;\ell) = E_\pm^{(m)}\psi_\pm^{(m)}(x;\ell), \tag{17}$$

where the eigenvalues are exactly constructed via WH algebra ladder relations and are given by

$$E_-^{(m)} = E_0 + 2m, \qquad E_+^{(m)} = E_0 + 2(m+1), \qquad m = 0, 1, 2, \ldots, \tag{18}$$

where E_0 is the energy of the Wigner oscillator ground state. Note that the energy spectrum of a particle in a potential given by bosonic sector Hamiltonian is equally spaced, similar to that of the 3D isotropic harmonic oscillator, with a difference of two quanta between two levels.

In addition, note that the operators $a^\pm(\ell+1)$ given in equation (4) together with $H(\ell+1)$, $J_\pm = (a^\pm(\ell+1))^2$ satisfy an osp(1 | 2) superalgebra.

3. The constrained super-Wigner oscillator in 4D

The usual isotropic oscillator in 4D has the following eigenvalue equation for its Hamiltonian H^B_{osc}, described by (employing the natural system of units $\hbar = m = 1$) the time-independent Schrödinger equation

$$H^B_{\mathrm{osc}}\Psi^B_{\mathrm{osc}}(y) = E^B_{\mathrm{osc}}\Psi^B_{\mathrm{osc}}(y), \tag{19}$$

with

$$H^B_{\mathrm{osc}} = -\tfrac{1}{2}\nabla_4^2 + \tfrac{1}{2}s^2, \qquad s^2 = \Sigma_{i=1}^4 y_i^2, \tag{20}$$

$$\nabla_4^2 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} + \frac{\partial^2}{\partial y_4^2} = \sum_{i=1}^{4}\frac{\partial^2}{\partial y_i^2}, \tag{21}$$

where the superscript B in H^B_{osc} is in anticipation of the Hamiltonian, with constraint to be defined, being implemented in the bosonic sector of the super $4D$ Wigner system with unitary frequency. Changing to spherical coordinates in four space dimensions and allowing a factorization of the energy eigenfunctions as a product of a radial eigenfunction and spin-spherical harmonic. In (21), the coordinates y_i $(i = 1, 2, 3, 4)$ in spherical coordinates in 4D are defined by [20, 23]

$$\begin{aligned} y_1 &= s\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\varphi-\omega}{2}\right) \\ y_2 &= s\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\varphi-\omega}{2}\right) \\ y_3 &= s\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\varphi+\omega}{2}\right) \\ y_4 &= s\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\varphi+\omega}{2}\right), \end{aligned} \tag{22}$$

where $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant \omega \leqslant 4\pi$.

The mapping of the coordinates $y_i (i = 1, 2, 3, 4)$ in 4D with the Cartesian coordinates ρ_i $(i = 1, 2, 3)$ in 3D is given by the Kustaanheimo–Stiefel transformation

$$\rho_i = \sum_{a,b=1}^{2} z_a^* \Gamma_{ab}^i z_b, \qquad (i = 1, 2, 3) \tag{23}$$

$$z_1 = y_1 + i y_2, \qquad z_2 = y_3 + i y_4, \tag{24}$$

where Γ_{ab}^i are the elements of the usual Pauli matrices. If one defines z_1 and z_2 as in equation (24), $Z = \binom{z_1}{z_2}$ is a two-dimensional spinor of SU(2) transforming as $Z \to Z' = UZ$ with U a two-by-two matrix of SU(2) and, of course, $Z^\dagger Z$ is invariant. So the transformation (23) is very spinorial. In addition, using the standard Euler angles in parametrizing SU(2) as in transformations (22) and (24), one obtains

$$z_1 = s \cos\left(\frac{\theta}{2}\right) \mathrm{e}^{\frac{\mathrm{i}}{2}(\varphi - \omega)} \qquad z_2 = s \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{\frac{\mathrm{i}}{2}(\varphi + \omega)}. \tag{25}$$

Note that the angles in these equations are divided by 2. However, in 3D, the angles are not divided by 2, namely $\rho_3 = \rho \cos^2\left(\frac{\theta}{2}\right) - \rho \sin^2\left(\frac{\theta}{2}\right) = \rho \cos\theta$. Indeed, from (23) and (25), we obtain

$$\rho_1 = \rho \sin\theta \cos\varphi, \qquad \rho_2 = \rho \sin\theta \sin\varphi, \qquad \rho_3 = \rho \cos\theta \tag{26}$$

and also that

$$\begin{aligned} \rho &= \left\{\rho_1^2 + \rho_2^2 + \rho_3^2\right\}^{\frac{1}{2}} = \left\{(\rho_1 + i\rho_2)(\rho_1 - i\rho_2) + \rho_3^2\right\}^{\frac{1}{2}} \\ &= \{(2z_1^* z_2)(2z_1 z_2^*) + (z_1^* z_1 - z_2^* z_2)^2\}^{\frac{1}{2}} \\ &= (z_1 z_1^* + z_2 z_2^*) = \sum_{i=1}^{4} y_i^2 = s^2. \end{aligned} \tag{27}$$

The complex form of the Kustaanheimo–Stiefel transformation was given by Cornish [21].

Thus, the expression for H_{osc}^B in (20) can be written in the form

$$\begin{aligned} H_{\mathrm{osc}}^B = &-\frac{1}{2}\left(\frac{\partial^2}{\partial s^2} + \frac{3}{s}\frac{\partial}{\partial s}\right) \\ &- \frac{2}{s^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin^2\theta}\left(2\cos\theta\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\omega}\right)\frac{\partial}{\partial\omega}\right] + \frac{1}{2}s^2. \end{aligned} \tag{28}$$

We obtain a constraint by projection (or 'dimensional reduction') from four to three dimensional. Note that ψ_{osc}^B is independent of ω and provides the constraint condition

$$\frac{\partial}{\partial\omega}\Psi_{\mathrm{osc}}^B(s, \theta, \varphi) = 0, \tag{29}$$

imposed on $H_{\mathrm{osc}}^{\mathrm{B}}$, the expression for this restricted Hamiltonian, which we continue to call $H_{\mathrm{osc}}^{\mathrm{B}}$, becomes

$$H_{\mathrm{osc}}^{\mathrm{B}} = -\frac{1}{2}\left(\frac{\partial^2}{\partial s^2} + \frac{3}{s}\frac{\partial}{\partial s}\right) - \frac{2}{s^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] + \frac{1}{2}s^2. \tag{30}$$

Identifying the expression in paranthesis in (30) with L^2, the square of the orbital angular momentum operator in 3D, since we always have

$$L^2 = (\vec{\sigma}\cdot\vec{L})(\vec{\sigma}\cdot\vec{L}+1), \tag{31}$$

which is valid for any system, where σ_i $(i = 1, 2, 3)$ are the Pauli matrices representing the spin $\frac{1}{2}$ degrees of freedom, we obtain for $H_{\rm osc}^B$ the final expression

$$H_{\rm osc}^B = \frac{1}{2}\left[-\left(\frac{\partial^2}{\partial s^2}+\frac{3}{s}\frac{\partial}{\partial s}\right)+\frac{4}{s^2}(\vec{\sigma}\cdot\vec{L})(\vec{\sigma}\cdot\vec{L}+1)+s^2\right]. \tag{32}$$

Now, associating $H_{\rm osc}^B$ with the bosonic sector of the super-Wigner system, $H_{\rm w}$, subject to the same constraint as in (29), and following the analogy with section 2 of construction of super-Wigner systems, we must first solve the Schrödinger equation

$$H_{\rm w}\Psi_{\rm w}(s,\theta,\varphi) = E_{\rm w}\Psi_{\rm w}(s,\theta,\varphi), \tag{33}$$

where the explicit form of $H_{\rm w}$ is given by

$$H_{\rm w}\left(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2}\right) = \begin{pmatrix} -\frac{1}{2}\left(\frac{\partial}{\partial s}+\frac{3}{2s}\right)^2+\frac{1}{2}s^2+\frac{(2\vec{\sigma}\cdot\vec{L}+\frac{1}{2})(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2})}{2s^2} & 0 \\ 0 & -\frac{1}{2}\left(\frac{\partial}{\partial s}+\frac{3}{2s}\right)^2+\frac{1}{2}s^2+\frac{(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2})(2\vec{\sigma}\cdot\vec{L}+\frac{5}{2})}{2s^2}\end{pmatrix}. \tag{34}$$

Using the operator technique in [2, 3], we begin with the following super-realized mutually adjoint operators:

$$a_{\rm w}^{\pm} \equiv a^{\pm}\left(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2}\right) = \frac{1}{\sqrt{2}}\left[\pm\left(\frac{\partial}{\partial s}+\frac{3}{2s}\right)\Sigma_1 \mp \frac{1}{s}\left(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2}\right)\Sigma_1\Sigma_3 - \Sigma_1 s\right], \tag{35}$$

where $\vec{\Sigma}_i$ $(i = 1, 2, 3)$ constitutes a set of Pauli matrices that provide the fermionic coordinates commuting with the similar Pauli set σ_i $(i = 1, 2, 3)$ already introduced representing the spin $\frac{1}{2}$ degrees of freedom.

It is checked, after some calculations, that a^+ and a^- of (35) are indeed the raising and lowering operators for the spectra of the super-Wigner Hamiltonian $H_{\rm w}$ respectively and they satisfy the following (anti-)commutation relations of the WH algebra:

$$\begin{aligned} H_{\rm w} &= \tfrac{1}{2}\left[a_{\rm w}^-, a_{\rm w}^+\right]_+ \\ &= a_{\rm w}^+a_{\rm w}^- + \tfrac{1}{2}\left[1+2\left(2\vec{\sigma}\cdot\vec{L}+\tfrac{3}{2}\right)\Sigma_3\right] \\ &= a_{\rm w}^-a_{\rm w}^+ - \tfrac{1}{2}\left[1+2\left(2\vec{\sigma}\cdot\vec{L}+\tfrac{3}{2}\right)\Sigma_3\right] \end{aligned} \tag{36}$$

$$\left[H_{\rm w}, a_{\rm w}^{\pm}\right]_- = \pm a_{\rm w}^{\pm} \tag{37}$$

$$\left[a_{\rm w}^-, a_{\rm w}^+\right]_- = 1+2\left(2\vec{\sigma}\cdot\vec{L}+\tfrac{3}{2}\right)\Sigma_3, \tag{38}$$

$$\left[\Sigma_3, a_{\rm w}^{\pm}\right]_+ = 0 \Rightarrow [\Sigma_3, H_{\rm w}]_- = 0. \tag{39}$$

Since the operator $\left(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2}\right)$ commutes with the basic elements $a^{\pm}$, Σ_3 and $H_{\rm w}$ of the WH algebra (36), (4) and (38) respectively, it can be replaced by its eigenvalues $\left(2\ell+\frac{3}{2}\right)$ and $-\left(2\ell+\frac{5}{2}\right)$ while acting on the respective eigenspace in the form

$$\Psi_{\rm osc}(s,\theta,\varphi) = \begin{pmatrix}\Psi_{\rm osc}^{\rm B}(s,\theta,\varphi)\\ \Psi_{\rm osc}^{\rm F}(s,\theta,\varphi)\end{pmatrix} = \begin{pmatrix}R_{\rm osc}^{\rm B}(s)\\ R_{\rm osc}^{\rm F}(s)\end{pmatrix} y_{\pm}(\theta,\varphi) \tag{40}$$

in the notation where $y_\pm(\theta,\varphi)$ are the spin-spherical harmonics [35, 36],

$$\begin{aligned} y_+(\theta,\varphi) &= y_{\ell\frac{1}{2};j=\ell+\frac{1}{2},m_j}(\theta,\varphi) \\ y_-(\theta,\varphi) &= y_{\ell+1\frac{1}{2};j=(\ell+1)-\frac{1}{2},m_j}(\theta,\varphi), \end{aligned} \tag{41}$$

so that we obtain $(\vec{\sigma}\cdot\vec{L}+1)y_\pm = \pm(\ell+1)y_\pm$, $\left(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2}\right)y_+ = \left(2\ell+\frac{3}{2}\right)y_+$ and $\left(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2}\right)y_- = -[2(\ell+1)+\frac{1}{2}]y_-$. Note that on these subspaces the 3D WH algebra is reduced to a formal 1D radial form with $H_{\rm w}\left(2\vec{\sigma}\cdot\vec{L}+\frac{3}{2}\right)$ acquiring respectively the forms $H_{\rm w}\left(2\ell+\frac{3}{2}\right)$ and

$$H_{\rm w}\left(-2\ell-\tfrac{5}{2}\right) = \Sigma_1 H_{\rm w}\left(2\ell+\tfrac{3}{2}\right)\Sigma_1. \tag{42}$$

Thus, the positive finite form of $H_{\rm w}$ in (36) together with the ladder relations (4) and the form (38) leads to the direct determination of the state energies and the corresponding Wigner ground-state wave functions by the simple application of the annihilation conditions

$$a^-\left(2\ell+\tfrac{3}{2}\right)\begin{pmatrix} R_{\rm osc}^{B^{(0)}}(s) \\ R_{\rm osc}^{F^{(0)}}(s)\end{pmatrix} = 0. \tag{43}$$

Then, the complete energy spectrum for $H_{\rm w}$ and the whole set of energy eigenfunctions $\Psi_{\rm osc}^{(n)}(s,\theta,\varphi)(n=2m,2m+1,m=0,2,\ldots)$ follow from the step-up operation provided by $a^+(2\ell+\frac{3}{2})$ acting on the ground state, which are also simultaneous eigenfunctions of the fermion number operator $N=\frac{1}{2}(1-\Sigma_3)$. We obtain for the bosonic sector Hamiltonian $H_{\rm osc}^B$ with the fermion number $n_f=0$ and even orbital angular momentum $\ell_4=2\ell$, $\ell=0,1,2,\ldots,$ the complete energy spectrum and eigenfunctions given by

$$\left[E_{\rm osc}^B\right]_{\ell_4=2\ell}^{(m)} = 2\ell+2+2m, \qquad (m=0,1,2,\ldots), \tag{44}$$

$$\left[\Psi_{\rm osc}^B(s,\theta,\varphi)\right]_{\ell_4=2\ell}^{(m)} \propto s^{2\ell}\exp\left(-\tfrac{1}{2}s^2\right)L_m^{(2\ell+1)}(s^2)\begin{cases} y_+(\theta,\varphi) \\ y_-(\theta,\varphi)\end{cases} \tag{45}$$

where $L_m^\alpha(s^2)$ are generalized Laguerre polynomials [2]. Now, to relate the mapping of the 4D super-Wigner system given by (8) with the corresponding system in 3D, we make use of the substitution of $s^2=\rho$, equation (29) and the following substitutions:

$$\frac{\partial}{\partial s} = 2\sqrt{\rho}\frac{\partial}{\partial\rho}, \qquad \frac{\partial^2}{\partial s^2} = 4\rho\frac{\partial^2}{\partial\rho^2}+2\frac{\partial}{\partial\rho}, \tag{46}$$

in (34) and divide the eigenvalue equation for $H_{\rm w}$ in (33) by $4s^2=4\rho$, obtaining

$$\begin{pmatrix} -\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial}{\partial\rho}\right)-\frac{1}{2}\left[-\frac{1}{4}-\frac{\vec{\sigma}\cdot\vec{L}(\vec{\sigma}\cdot\vec{L}+1)}{\rho^2}\right] & 0 \\ 0 & -\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial}{\partial\rho}\right)-\frac{1}{2}\left[-\frac{1}{4}-\frac{(\vec{\sigma}\cdot\vec{L}+\frac{1}{2})(\vec{\sigma}\cdot\vec{L}+\frac{3}{2})}{\rho^2}\right]\end{pmatrix}\begin{pmatrix}\Psi^B \\ \Psi^F\end{pmatrix}$$
$$= \frac{1}{4\rho}E_{\rm w}\begin{pmatrix}\Psi^B \\ \Psi^F\end{pmatrix}. \tag{47}$$

The bosonic sector of the above eigenvalue equation can immediately be identified with the eigenvalue equation for the Hamiltonian of the 3D hydrogen-like atom expressed in the equivalent form given by

$$\left\{-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial}{\partial\rho}\right)-\frac{1}{2}\left[-\frac{1}{4}-\frac{\vec{\sigma}\cdot\vec{L}(\vec{\sigma}\cdot\vec{L}+1)}{\rho^2}\right]\right\}\psi(\rho,\theta,\varphi) = \frac{\lambda}{2\rho}\psi(\rho,\theta,\varphi), \tag{48}$$

where $\Psi^B = \psi(\rho, \theta, \varphi)$ and the connection between the dimensionless and dimensional eigenvalues, respectively, λ and E_a with $e = 1 = m = \hbar$ is given by [36]

$$\lambda = \frac{Z}{\sqrt{-2E_a}}, \qquad \rho = \alpha r, \qquad \alpha = \sqrt{-8E_a}, \tag{49}$$

where E_a is the energy of the electron hydrogen-like atom and (r, θ, φ) stand for the spherical polar coordinates of the position vector $\vec{r} = (x_1, x_2, x_3)$ of the electron in relation to the nucleons of charge Z together with $s^2 = \rho$. We see then from equations (44), (45), (48) and (49) that the complete energy spectrum and eigenfunctions for the hydrogen-like atom are given by

$$\frac{\lambda}{2} = \frac{E^B_{\text{osc}}}{4} \Rightarrow [E_a]^{(m)}_{\ell} = [E_a]^{(N)} = -\frac{Z^2}{2N^2}, \qquad (N = 1, 2, \ldots), \tag{50}$$

and

$$[\psi(\rho, \theta, \varphi)]^{(m)}_{\ell;,m_j} \propto \rho^{\ell} \exp\left(-\frac{\rho}{2}\right) L^{(2\ell+1)}_m(\rho) \begin{cases} y_+(\theta, \varphi) \\ y_-(\theta, \varphi) \end{cases} \tag{51}$$

where E^B_{osc} is given by equation (44).

Here, $N = \ell + m + 1 (\ell = 0, 1, 2, \ldots, N - 1; m = 0, 1, 2, \ldots)$ is the principal quantum number. Kostelecky and Nieto have shown that the supersymmetry in non-relativistic quantum mechanics may be realized in atomic systems [25].

4. Conclusion

In this work, we have deduced the energy eigenvalues and eigenfunctions of the hydrogen atom via the Wigner–Heisenberg (WH) algebra in non-relativistic quantum mechanics. Indeed, from the ladder operators for the four-dimensional (4D) super-Wigner system, ladder operators for the mapped super 3D system, and hence for the hydrogen-like atom in bosonic sector, are deduced. The complete spectrum for the hydrogen atom is found with considerable simplicity. Therefore, the solutions of the time-independent Schrödinger equation for the hydrogen atom were mapped onto the super-Wigner harmonic oscillator in 4D using the Kustaanheimo–Stiefel transformation.

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